Inverse of a Matrix

If A and B are two square matrices such that $AB = I_n = BA$ then B is called the inverse of A and A is called Inverse of B. The inverse of A is denoted by A^{-1} and $AA^{-1} = I_n = A^{-1}A$

If the inverse of a square matrix A exists then A is called Invertible Matrix

Theorem: Uniqueness of Inverse

Inverse of a square matrix, if it exist, is unique.

Inverse of matrix using adjoint

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A \quad ; |A| \neq 0$$

A square matrix is invertible iff A is a non-singular matrix.

* Find the inverse of a matrix $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$, if it exists.

Sol. Given $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

Now, $|A| = -1 \times 2 - 5 \times -3 = -2 + 15 = 13 \neq 0$

$\therefore A^{-1}$ exists.

Now,

$$M_{11} = 2 \quad , \quad C_{11} = (-1)^{1+1} M_{11} = 2$$

$$M_{12} = -3 \quad , \quad C_{12} = (-1)^{1+2} M_{12} = 3$$

$$M_{21} = 5 \quad , \quad C_{21} = (-1)^{2+1} M_{21} = -5$$

$$M_{22} = -1 \quad , \quad C_{22} = (-1)^{2+2} M_{22} = -1$$

Therefore, the Cofactor matrix,

$$C = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$$

Now, $\text{adj}A = C^T = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

Therefore,

$$A^{-1} = \frac{1}{|A|} \text{adj} A$$

$$= \frac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

* Find the inverse of a matrix, $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$

Sol: $|A| = 3(2-3) - -2(4--4) + 3(-6-4)$

$= 3 \times -1 + 2 \times 8 + 3 \times -10$

$= -17 \neq 0$, hence A^{-1} exists.

$M_{11} = 2 - 3 = -1$, $C_{11} = -1$

$M_{12} = 4 - -4 = 8$, $C_{12} = -8$

$M_{13} = -6 - 4 = -10$, $C_{13} = -10$

$M_{21} = -4 - -9 = 5$, $C_{21} = -5$

$M_{22} = 6 - 12 = -6$, $C_{22} = -6$

$M_{23} = -9 - -8 = -1$, $C_{23} = 1$

$M_{31} = 2 - 3 = -1$, $C_{31} = -1$

$M_{32} = -3 - 6 = -9$, $C_{32} = 9$

$M_{33} = 3 - -4 = 7$, $C_{33} = 7$

Therefore, the cofactor matrix,

$$C = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}$$

$$\text{Adj} A = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

Therefore,

$$A^{-1} = -1/17 \begin{bmatrix} -1 & -5 & 1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

Theorem :

If A and B are two invertible matices of same order then,

$$(AB)^{-1} = B^{-1}.A^{-1}$$

* Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

verify that $(AB)^{-1} = B^{-1}.A^{-1}$

Sol: $LHS = (AB)^{-1}$

$$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 18+49 & 24+63 \\ 18+35 & 16+45 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

$$|AB| = 67 \times 61 - 87 \times 47$$

$$= 4087 - 4089 = -2$$

$M_{11} = 61$, $C_{11} = 61$

$M_{12} = 47$, $C_{12} = -47$

$M_{21} = 87$, $C_{21} = -87$

$M_{22} = 67$, $C_{22} = 67$

$$Adj(AB) = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

Now,

$$(AB)^{-1} = \frac{1}{|AB|} adj(AB)$$

$$= \frac{-1}{2} \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \quad \text{—} (1)$$

$RHS = B^{-1} . A^{-1}$

$$Adj\,B = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} , \quad Adj\,A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$|A| = 3\times5 - 2\times7 = 15-14 = 1$

$|B| = 6\times9 - 7\times8 = 54-56 = -2$

$$B^{-1} = \frac{-1}{2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} , \quad A^{-1} = 1 \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

Now,

$$B^{-1} \cdot A^{-1} = \frac{-1}{9}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{-1}{9}\begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix}$$

$$= \frac{-1}{9}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \quad \text{—} (2)$$

From (1) & (2), $(AB)^{-1} = B^{-1} \cdot A^{-1}$ hence verified.

Note: If A and B are square matrices of same order then $|AB| = |A||B|$

Properties:

1. If A is a square matrix then,

$$(\text{adj} A)^T = \text{adj}(A^T)$$

2. If A is an invertible matrix then,

$$(A^T)^{-1} = (A^{-1})^T$$

$$(\text{adj} A)^{-1} = \text{adj}(A^{-1})$$

$$(A^{-1})^{-1} = A$$